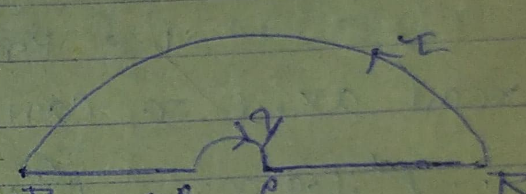
M.N: M: 30.96

Q No → By integrating $\frac{(\log z)^2}{1+z^2}$ round a suitable Contour, Prove that

$$\int_0^\infty \frac{(\log x)^2 dx}{1+x^2} dx = \frac{\pi^3}{8}.$$

Soln:- Let us Consider the integral

$$\int_C \frac{(\log z)^2}{1+z^2} dz = \int_C f(z) dz$$



Where, C is Contour Consisting of the real axis from −R to −ρ the Semi Circle γ of small radius ρ the real axis from ρ to R and the Semi Circle Γ of large radius R

$$\therefore \int_C f(z) dz = \int_{-R}^{-\rho} f(x) dx + \int_\gamma f(z) dz + \int_\rho^R f(x) dx$$

$$+ \int_\Gamma f(z) dz = 2\pi i \Sigma R^+.$$

$$\text{Since, } \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z(\log z)^2}{z^2+1} = 0$$

$$\text{We have, } \lim_{R\to\infty} \int_\Gamma f(z) dz = 0$$

$$\text{Again, } \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z(\log z)^2}{z^2+1}$$

$$= \underset{z\to\infty}{Lt.} \frac{(\log z)^2}{1/z} = \underset{z\to\infty}{Lt} \frac{2\log z}{z\times\frac{1}{z^2}}$$

$$= \underset{z\to 0}{Lt.} \frac{2}{z\times\frac{1}{z^2}} = \underset{z\to 0}{Lt}\, 2z = 0,$$ we have

$$\underset{\rho\to 0}{Lt.} \int_{\gamma} f(z)\,dz = 0$$

$$\therefore \int_{\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

Since on the $-ve$ part of the real axis x has the value $xe^{i\pi}$, we have

$$\int_{-\infty}^{0} f(x)\,dx = \int_{\infty}^{0} f(xe^{i\pi})e^{i\pi}dx = \int_{0}^{\infty} f(xe^{i\pi})\,dx$$

$$\therefore \int_{0}^{\infty} f(xe^{i\pi})\,dx + \int_{0}^{\infty} f(x)\,dx = 2\pi i \sum R^+.$$

The Poles of $f(z)$ are given by $z^2+1=0$, i.e. $z=\pm i$ of which only $z=i$ lies within C.

$$\therefore \sum R^+ = \underset{z\to i}{Lt.}(z-i)f(z) = \underset{z\to i}{Lt}\frac{(z-i)(\log z)^2}{z^2+1}$$

$$= \underset{z\to i}{Lt.}\frac{(\log z)^2}{z+i} = \frac{1}{2i}(\log e^{i\pi/2})^2$$

$$= \frac{1}{2i}\left(\frac{i\pi}{2}\right)^2 = -\frac{\pi^2}{8i}$$

$$\therefore \int_0^\infty f(xe^{i\pi})\,dx + \int_0^\infty f(x)\,dx = 2\pi i \Sigma R^+$$

$$= 2\pi i \times -\frac{\pi^2}{\cancel{8}_4 i} = -\frac{\pi^3}{4}.$$

$$\therefore \int_0^\infty \frac{(\log xe^{i\pi})^2}{1+x^2}\,dx + \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = -\frac{\pi^3}{4}$$

We have $(\log x e^{i\pi})^2 = (\log x + \log e^{i\pi})^2$

$$= (\log x + i\pi)^2 = (\log x)^2 + 2i\pi \log x - \pi^2$$

$$\therefore \int_0^\infty \frac{(\log x)^2 + 2i\pi \log x - \pi^2}{1+x^2}\,dx + \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = -\frac{\pi^3}{4}$$

Equating the real and imaginary parts both sides, we have

$$2\int_0^\infty \frac{(\log x)^2\,dx}{1+x^2} - \pi^2 \int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi^3}{4} \text{ and, } \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$$

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = -\frac{\pi^3}{4} + \pi^2 \int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi^3}{4} + \pi^2 (\tan^{-1} x)_0^\infty$$

$$= -\frac{\pi^3}{4} + \pi^2 (\tan^{-1}\infty - \tan^{-1} 0)$$

$$= -\frac{\pi^3}{4} + \pi^2 \cdot \frac{\pi}{2} = \frac{\pi^3}{4}$$

$$\therefore \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}$$ Proved.

QNo→ Prove by Contour integration

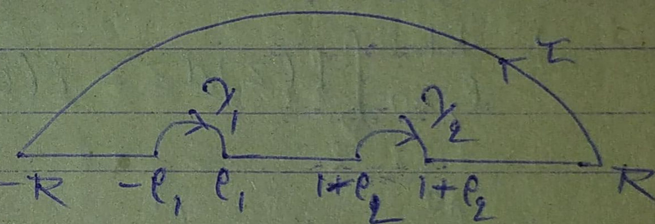
$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi} \quad (0<a<1)$$

and $$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \pi \cot a\pi \quad (0<a<1).$$

Soln.:→ We Consider the integral

$$\int_C \frac{z^{a-1}}{1-z}\,dz = \int_C f(z)\,dz$$

Where C is Contour



Consisting of real axis $-R$ to $-\ell_1$ the Semi circle γ_1 of Small radius ℓ_1 the real axis from ℓ_1 to $1-\ell_1$. the Semi Circle γ_2 of Small radius ℓ_2 the real axis from $1+\ell_2$ to ~~$1-\ell_2$~~ R together with the Semi circle Γ of large radius R.

$$\therefore \int_C f(z)\,dz = \int_{-R}^{-\ell_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{\ell_1}^{1-\ell_2} f(x)\,dx$$

$$+ \int_{\gamma_2} f(z)\,dz + \int_{1+\ell_2}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 0$$

Since, $f(z)$ has no Singularities within C, we have

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z \cdot z^{a-1}}{1-z} = \lim_{z\to\infty} \frac{z^a}{1-z} = 0$$

$$\therefore \underset{R\to 0}{Lt} \int_{\tau} f(z)\,dz = i(\pi - 0)\cdot 0 = 0$$

$$\because \underset{z\to 0}{Lt}\; z f(z) = \underset{z\to 0}{Lt}\; \frac{z\cdot z^{a-1}}{1-z} = \underset{z\to 0}{Lt}\left(\frac{z^{a}}{1-z}\right) = 0$$

$$\therefore \underset{\rho_1\to 0}{Lt} \int_{\gamma_1} f(z)\,dz = -i(\pi - 0)\cdot 0 = 0$$

$$\because \underset{z\to 1}{Lt}\,(z-1) f(z) = \underset{z\to 1}{Lt}\; \frac{(z-1)z^{a-1}}{1-z} = -1$$

$$\therefore \underset{\rho_2\to 0}{Lt} \int_{\gamma_2} f(z)\,dz = i(0-\pi)(-1) = i\pi .$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx + i\pi = 0$$

On the -ve part of the real axis x has the value $xe^{i\pi}$.

$$\therefore \int_{-\infty}^{0} f(x)\,dx = \int_{\infty}^{0} f(xe^{i\pi})\,e^{i\pi}\,dx = \int_{0}^{\infty} f(xe^{i\pi})\,dx .$$

$$\therefore \int_{0}^{\infty} f(xe^{i\pi})\,dx + \int_{0}^{\infty} f(x)\,dx = -i\pi .$$

$$\therefore \int_{0}^{\infty} \frac{x^{a-1}e^{i\pi(a-1)}}{1-xe^{i\pi}} + \int_{0}^{\infty} \frac{x^{a-1}}{1-x}\,dx = -i\pi .$$

i.e. $$\int_{0}^{\infty} \frac{x^{a-1}(\cos\pi a + i\sin\pi a)}{1+x}\,dx + \int_{0}^{\infty} \frac{x^{a-1}}{1-x}\,dx = -i\pi$$

Equating the real and imaginary parts, we have

$$-\int_0^\infty \frac{x^{a-1}\cos a\pi \, dx}{1+x} + \int_0^\infty \frac{x^{a-1}}{1-x}\, dx = 0$$

and, $$-\int_0^\infty \frac{x^{a-1}\sin a\pi}{1+x}\, dx = -\pi$$

$$\therefore \int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin a\pi}$$

$$\boxed{\begin{aligned} &\int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \int_0^\infty \frac{x^{a-1}}{1-x}\, dx \\ &\therefore \int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \frac{\pi \cos a\pi}{\sin a\pi} = \pi \cot a\pi \end{aligned}}$$

and, $$\int_0^\infty \frac{x^{a-1}}{1 \cancel{+} x}\, dx = \cos a\pi \int_0^\infty \frac{x^{a-1}}{1+x}\, dx$$

$$= \cos a\pi \cdot \frac{\pi}{\sin a\pi} = \pi \cot a\pi$$

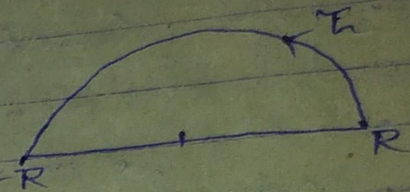
$$\therefore \int_0^\infty \frac{x^{a-1}}{1-x}\, dx = \pi \cot a\pi \cdot \text{Proved}$$

M.U. 90

Q No.- Prove that $\int_0^{\infty} \frac{\cos x^2 + \sin x^2 - 1}{x^2}\, dx = 0$

Soln. Consider the integral

$$\int_C \frac{e^{iz^2} - 1}{z^2}\, dz = \int_C f(z)\, dz.$$



Where C is Contour consisting of the real axis from $-R$ to R and the semi circle Γ of large radius R in the upper half of the Z-Plane.

$$\therefore \int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 0$$

Let $z = Re^{i\theta}$, $dz = Rie^{i\theta}\, d\theta$

$$\therefore \int_{\Gamma} f(z)\, dz = \int_0^{\pi} \frac{e^{iR^2 e^{2i\theta}} - 1}{R^2 e^{2i\theta}} \cdot Rie^{i\theta}\, d\theta$$

$$= \frac{i}{R} \int_0^{\pi} e^{-i\theta} \left[\exp\{ iR^2 (\cos 2\theta + i \sin 2\theta)\} - 1 \right] d\theta$$

$$= \int_0^{\pi} \frac{i}{R} e^{-i\theta} \left[\{\exp(-R^2 \sin 2\theta)\} \times \{\exp(iR^2 \cos 2\theta)\} - 1 \right] d\theta$$

$$\therefore \left| \int_{\Gamma} f(z)\, dz \right| \leq \frac{1}{R} \int_0^{\pi} |ie^{-i\theta}| \times [\exp(-R^2 \sin 2\theta) |\exp(iR^2 \cos 2\theta)| + |-1|]\, d\theta$$

$$\leq \frac{1}{R} \int_0^{\pi} [\exp(-R^2 \sin 2\theta) + 1)]\, d\theta$$

$\rightarrow 0$ as $R \rightarrow \infty$.

Hence, when $R \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 0$$

or, $$\int_{-\infty}^{\infty} \frac{e^{ix^2}-1}{x^2}\,dx = 0$$ or, $$2\int_0^{\infty} \frac{e^{ix^2}-1}{x^2}\,dx = 0$$

Equating real and imaginary Parts,

$$\int_0^{\infty} \frac{\cos x^2 - 1}{x^2}\,dx = 0 \quad \text{(A)} \quad \& \quad \int_0^{\infty} \frac{\sin x^2}{x^2}\,dx = 0 \quad \text{(B)}$$

Adding (A) & (B), we get

$$\int_0^{\infty} \frac{\cos x^2 + \sin x^2 - 1}{x^2}\,dx = 0.$$